Ordered Interpretations and Entailment for Defeasible Description Logics

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Abstract. We enrich description logics with non-monotonic reasoning features. We start by investigating a notion of *defeasible subsumption relation* in the spirit of KLM-style defeasible consequence. In particular we provide a natural and intuitive semantics for defeasible subsumption in terms of DL interpretations enriched with a preference relation. We propose syntactic characterizations for both *preferential* and *rational* subsumption relations (in terms of Gentzen-style rules or postulates) and prove representation results for the description logic $\mathcal{ALC}$. We then move to non-monotonicity in DLs at the level of *entailment*. We investigate versions of entailment in the context of both preferential and rational subsumption, relate them to preferential and rational closure, and show that computing them can be reduced to classical $\mathcal{ALC}$ entailment, providing further evidence that our semantic constructions are appropriate in a non-monotonic DL setting.

Keywords: Defeasible reasoning; description logics; preferential semantics; rational closure

1 Introduction

Description logics (DLs) [1] are central to many modern AI applications because they provide the logical foundation of formal ontologies. Endowing DLs with non-monotonic features is therefore a problem of paramount importance from the standpoint of knowledge representation and reasoning. Indeed, the past 20 years have witnessed many attempts to introduce non-monotonicity in a DL setting, mostly ranging from preferential approaches [27, 9, 19, 14] to circumscription [6], amongst others [2, 17].

Preferential extensions of DLs turn out to be particularly promising mostly because they are based on one of the most comprehensive and successful frameworks for non-monotonic reasoning in the propositional case, namely the *KLM approach* [22, 26]. The success of the KLM approach is due to a number of reasons. Firstly, it provides a thorough analysis of the properties that any non-monotonic consequence relation deemed 'well-behaved' is supposed to satisfy, which plays a central role in assessing how intuitive the obtained results are; secondly, it allows for many decision problems to be reduced to classical entailment checking, sometimes without blowing up the computational complexity with respect to the classical case, and, thirdly, it has a well-known connection with belief revision [18, 21]. It is reasonable to expect that most of these features will transfer to KLM-based extensions of DLs.

It turns out that existing extensions of the KLM approach have mostly been driven by either extending only the syntax [19, 14] or just the underlying preferential semantics [25, 27, 9]. A comprehensive study of defeasible reasoning in DLs, with an intuitive *semantics*, a corresponding *representation result*, and an appropriate notion of *entailment* is still to see the light of day.

In this paper we fill this gap. We take the following route: After fixing the notation (Section 2) we present the notion of defeasible subsumption *à la* KLM (Section 3). In particular, we give an intuitive semantics for the idea that "usually C is subsumed by D" and we provide a characterization (via representation results) of two important classes of defeasible subsumption relations, namely preferential and rational subsumption. In Section 4 we define appropriate notion of entailment (*preferential and minimal ranked entailment*) for preferential and ranked interpretations. We explore the relationship that minimal ranked entailment has with both Lehmann and Magidor's [26] definition of rational closure and Casini and Straccia's [14] algorithm for its computation (Section 5). After a discussion of and comparison with related work (Section 6), we conclude with a summary of our contributions and directions for further exploration.

2 Preliminaries

The language of the description logic $\mathcal{ALC}$ is built up from a finite set of *concept names* $\mathsf{N}_{\mathscr{C}}$ and a finite set of *role names* $\mathsf{N}_{\mathscr{R}}$ such that $\mathsf{N}_{\mathscr{C}} \cap \mathsf{N}_{\mathscr{R}} = \emptyset$. A concept name is denoted by A and a role name by r. Complex concepts are denoted by $C, D, \ldots$, possibly with subscripts, and are built in the usual way according to the rule:

$$C ::= A \mid \neg C \mid C \sqcap C \mid \exists r.C$$

Concepts built with the constructors $\sqcup$ and $\forall$ are defined in terms of the others in the usual way. We use $\top$ as an abbreviation for $A \sqcup \neg A$ and $\bot$ as an abbreviation for $A \sqcap \neg A$, for some $A \in \mathsf{N}_{\mathscr{C}}$. We denote the set of all $\mathcal{ALC}$ concepts by $\mathcal{L}$.

The semantics of $\mathcal{ALC}$ is the standard set theoretic Tarskian semantics. An *interpretation* is a structure $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set called the *domain*, and $\cdot^{\mathcal{I}}$ is an *interpretation function* mapping concept names A in $\mathsf{N}_{\mathscr{C}}$ to subsets $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, and mapping role names r in $\mathsf{N}_{\mathscr{R}}$ to binary relations $r^{\mathcal{I}}$ over $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$: $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}, r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. Given an interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, $\cdot^{\mathcal{I}}$ is extended to interpret complex concepts in the following way:

$$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}, \quad (C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}},$$

$$(\exists r.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \text{for some } y, (x, y) \in r^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}$$

Given $C, D \in \mathcal{L}$, $C \sqsubseteq D$ is a *subsumption statement.* $C \equiv D$ is an abbreviation for both $C \sqsubseteq D$ and $D \sqsubseteq C$. We denote a subsumption statement with α.

An $\mathcal{ALC}$ *TBox* $\mathcal{T}$ is a finite set of subsumption statements. An interpretation $\mathcal{I}$ *satisfies* $C \sqsubseteq D$ (denoted $\mathcal{I} \Vdash C \sqsubseteq D$) if and only if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. α is (classically) *entailed* by a TBox $\mathcal{T}$, denoted $\mathcal{T} \models \alpha$, if and only if $\mathcal{I} \Vdash \alpha$ for every $\mathcal{I}$ such that $\mathcal{I} \Vdash \beta$ for all $\beta \in \mathcal{T}$. For more details on DLs the reader is referred to the Description Logic handbook [1].

3 Defeasible Subsumption

In this section we present our semantics for preferential and rational subsumption by enriching standard DL interpretations $\mathcal{I}$ with an ordering on the elements of the domain $\Delta^{\mathcal{I}}$. The intuition underlying it is simple and natural, and extends similar work done for the propositional case [22, 26]. Variants of the approach we take have been proposed as well [8, 3, 9, 11, 20]. However, this is the first comprehensive semantic account of preferential and rational subsumption based on the standard semantics for DLs.

Informally, the semantics is based on the idea that objects of the domain can be ordered according to their degree of *normality* [8] or *typicality* [9, 20, 7]. We do not require that there exist something intrinsic about objects that makes one object more normal than another. Rather, the intention is to provide a framework in which to express all conceivable ways in which objects, with their associated properties and relationships with other objects, can be ordered in terms of typicality, in the same way that the class of all DL standard interpretations constitute a framework representing all conceivable ways of representing the properties of objects and their relationships with other objects.

Definition 1 (Preferential Interpretation). *A preferential interpretation is a structure $\mathcal{P} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec_{\mathcal{P}} \rangle$, where $\langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}} \rangle$ is a DL interpretation (which we denote by $\mathcal{I}_{\mathcal{P}}$ and refer to as the interpretation associated with $\mathcal{P}$), and $\prec_{\mathcal{P}}$ is a strict partial order on $\Delta^{\mathcal{P}}$ (i.e., $\prec_{\mathcal{P}}$ is irreflexive and transitive) satisfying the smoothness condition (for every $C \in \mathcal{L}$, $\min_{\prec_{\mathcal{P}}}(C^{\mathcal{P}}) \neq \emptyset$).*

As an example of a preferential interpretation, let $\mathsf{N}_{\mathscr{C}} = \{A, B\}$ and let $\mathsf{N}_{\mathscr{R}} = \{r\}$. Figure 1 below depicts the preferential interpretation $\mathcal{P} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec_{\mathcal{P}} \rangle$, where $\Delta^{\mathcal{P}} = \{x_i \mid 1 \leq i \leq 5\}$, $A^{\mathcal{P}} = \{x_1, x_2, x_3\}$, $B^{\mathcal{P}} = \{x_2, x_3, x_4\}$, $r^{\mathcal{P}} = \{(x_1, x_2), (x_2, x_3), (x_3, x_2), (x_1, x_4), (x_4, x_5), (x_5, x_4)\}$, which is represented by the solid arrows in the picture, and $\prec_{\mathcal{P}}$ is the transitive closure of $\{(x_1, x_2), (x_1, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5)\}$, i.e., of the relation represented by the dashed arrows in the picture. (Note the direction of the dashed arrows, pointing from less to more preferred objects, with more preferred objects lower in the order.)

A preferential interpretation $\mathcal{P}$ *satisfies* a (classical) subsumption statement $C \sqsubseteq D$ (denoted $\mathcal{P} \Vdash C \sqsubseteq D$) if and only if $C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$. It is easy to see that the addition of the $\prec_{\mathcal{P}}$-component preserves the truth of all subsumption statements holding in the remaining structure:

Lemma 1. *Let $\mathcal{P} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec_{\mathcal{P}} \rangle$ be a preferential interpretation. For every α, $\mathcal{P} \Vdash \alpha$ if and only if $\mathcal{I}_{\mathcal{P}} \Vdash \alpha$.*

$\mathcal{P}$:

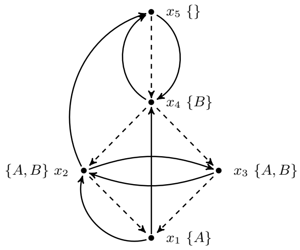

Fig. 1. A preferential interpretation.

Given $C, D \in \mathcal{L}$, a statement of the form $C \sqsubsetsim D$ is a *defeasible subsumption statement* and is read "*usually* C is subsumed by D". Paraphrasing Lehmann [23], the intuition of $C \sqsubsetsim D$ is that "if C were all the information about an object available to an agent, then D would be a sensible conclusion to draw about such an object". Note that $\sqsubsetsim$ is a connective on the object level and is meant to be the defeasible counterpart of $\sqsubseteq$. A preferential interpretation $\mathcal{P} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec_{\mathcal{P}} \rangle$ satisfies a defeasible subsumption statement $C \sqsubsetsim D$, denoted by $\mathcal{P} \Vdash C \sqsubsetsim D$, if and only if $\min_{\prec_{\mathcal{P}}}(C^{\mathcal{P}}) \subseteq D^{\mathcal{P}}$.

As an example, in the interpretation $\mathcal{P}$ of Figure 1, we have $\mathcal{P} \Vdash A \sqsubsetsim \forall r.B$ (but note that $\mathcal{P} \not\Vdash A \sqsubseteq \forall r.B$).

In additon to preferential interpretations we also study *ranked* interpretations, i.e., preferential interpretations in which the $\prec$-component is a *modular* ordering:

Definition 2 (Modular Order). *Given a set X, $\prec \subseteq X \times X$ is modular if and only if there is a ranking function $rk : X \longrightarrow \mathbb{N}$ s.t. for every $x, y \in X$, $x \prec y$ iff $rk(x) < rk(y)$.*

Definition 3 (Ranked Interpretation). *A ranked interpretation is a preferential interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_{\mathcal{R}} \rangle$ such that $\prec_{\mathcal{R}}$ is modular.*

It turns out that the preferential interpretation in Figure 1 is also an example of a ranked interpretation.

It is worth noting that every (classical) subsumption statement is equivalent, with respect to preferential interpretations (and therefore also with respect to ranked interpretations), to a defeasible subsumption statement.

Lemma 2. *For every preferential interpretation $\mathcal{P}$, and every $C, D \in \mathcal{L}$, $\mathcal{P} \Vdash C \sqsubseteq D$ if and only if $\mathcal{P} \Vdash C \sqcap \neg D \sqsubsetsim \bot$.*

For the remainder of this paper we therefore only consider defeasible subsumption statements, and regard any classical subsumption statement $C \sqsubseteq D$ as an abbreviation of the defeasible subsumption statement $C \sqcap \neg D \sqsubsetsim \bot$.

In order to study the properties of defeasible subsumption it is useful to consider the following class of binary relations:

Definition 4. *A defeasible subsumption relation is a binary relation $\leadsto \subseteq \mathcal{L} \times \mathcal{L}$.*

Given a defeasible subsumption relation $\rightsquigarrow$, we say that $\rightsquigarrow$ is *preferential* if it satisfies the following set of properties which we refer to as (the DL versions of the) *preferential* KLM properties:

$$\text{(Cons) } \top \not\rightsquigarrow \bot \quad \text{(Ref) } C \rightsquigarrow C \quad \text{(LLE) } \frac{\models C \equiv D,\ C \rightsquigarrow E}{D \rightsquigarrow E}$$

$$\text{(And) } \frac{C \rightsquigarrow D,\ C \rightsquigarrow E}{C \rightsquigarrow D \sqcap E} \qquad \text{(Or) } \frac{C \rightsquigarrow E,\ D \rightsquigarrow E}{C \sqcup D \rightsquigarrow E}$$

$$\text{(RW) } \frac{C \rightsquigarrow D,\ \models D \sqsubseteq E}{C \rightsquigarrow E} \qquad \text{(CM) } \frac{C \rightsquigarrow D,\ C \rightsquigarrow E}{C \sqcap D \rightsquigarrow E}$$

The last six properties are the obvious translations of the properties for preferential consequence relations proposed by Kraus et al. [22]. They have been discussed at length in the literature for both the propositional and the DL cases [22, 26, 24, 19] and we shall not do so here. Property Cons is the DL version of the Consistency property usually assumed when linking KLM defeasible consequence with belief revision [18]. It corresponds to the requirement that preferential interpretations, like standard DL interpretations, have non-empty domains.

Definition 5. *Let* $\rightsquigarrow_{\mathcal{P}} := \{(C, D) \mid \mathcal{P} \Vdash C \,\tilde{\sqsubseteq}\, D\}$ *be the defeasible subsumption relation induced by* $\mathcal{P}$*.*

The first important result we present shows that the defeasible subsumption relations induced by preferential interpretations are precisely the preferential subsumption relations.

Theorem 1. *A defeasible subsumption relation* $\rightsquigarrow \subseteq \mathcal{L} \times \mathcal{L}$ *is preferential if and only if there is a preferential interpretation* $\mathcal{P}$ *such that* $\rightsquigarrow_{\mathcal{P}} = \rightsquigarrow$*.*

What is perhaps surprising about this result is that no additional properties based on the structure of $\mathcal{ALC}$ are necessary to characterize the subsumption relations induced by preferential interpretations. We provide below a number of properties involving the use of quantifiers that are satisfied by all preferential subsumption relations.

The first two are 'existential' and 'universal' versions of cautious monotonicity (CM):

$$(\text{CM}_\exists) \ \frac{\exists r.C \rightsquigarrow E,\ \exists r.C \rightsquigarrow \forall r.D}{\exists r.(C \sqcap D) \rightsquigarrow E}$$

$$(\text{CM}_\forall) \ \frac{\forall r.C \rightsquigarrow E,\ \forall r.C \rightsquigarrow \forall r.D}{\forall r.(C \sqcap D) \rightsquigarrow E}$$

The third one is similar to the Rule of Necessitation in modal logic [16]:

$$\text{(Norm) } \frac{C \rightsquigarrow \bot}{\exists r.C \rightsquigarrow \bot}$$

If, in addition to the preferential properties, the relation $\rightsquigarrow$ also satisfies rational monotonicity (RM) below, it is said to be a *rational* subsumption relation:

$$\text{(RM) } \frac{C \rightsquigarrow E,\ C \not\rightsquigarrow \neg D}{C \sqcap D \rightsquigarrow E}$$

When considering rational subsumption relations one has to move to *ranked* interpretations. This brings us to our second important result, showing that the defeasible subsumption relations induced by ranked interpretations are precisely the rational subsumption relations.

Theorem 2. *A defeasible subsumption relation* $\rightsquigarrow \subseteq \mathcal{L} \times \mathcal{L}$ *is rational if and only if there is a ranked interpretation* $\mathcal{R}$ *such that* $\rightsquigarrow_{\mathcal{R}} = \rightsquigarrow$.

Analogous to the case for cautious monotonicity, the following 'existential' and 'universal' versions of rational monotonicity are satisfied by all rational subsumption relations:

$$(\mathrm{RM}_{\exists})\ \frac{\exists r.C \rightsquigarrow E,\ \exists r.C \not\rightsquigarrow \forall r.\neg D}{\exists r.(C \sqcap D) \rightsquigarrow E}$$

$$(\mathrm{RM}_{\forall})\ \frac{\forall r.C \rightsquigarrow E,\ \forall r.C \not\rightsquigarrow \forall r.\neg D}{\forall r.(C \sqcap D) \rightsquigarrow E}$$

4 DBox Entailment

Given a (finite) set of defeasible subsumption statements of the form $C \precsim D$, from a knowledge representation and reasoning perspective it becomes important to address the question of what it means for a defeasible subsumption statement to be *entailed* by others.

Definition 6 (DBox). *A defeasible TBox (or DBox)* $\mathcal{D}$ *is a finite set of defeasible subsumption statements.*

Observe that, since every classical subsumption statement can be viewed as an abbreviation of a defeasible subsumption statement (Lemma 2), it follows that a classical TBox can be viewed as a special case of a DBox. We use $\alpha, \beta, \ldots$ to denote elements of a DBox.

We begin our investigation of DBox entailment in the context of preferential interpretations. Given a preferential interpretation $\mathcal{P}$ and a DBox $\mathcal{D}$, we extend the notion of satisfaction to DBoxes in the obvious way: $\mathcal{P}$ satisfies $\mathcal{D}$ (noted $\mathcal{P} \Vdash \mathcal{D}$), if and only if $\mathcal{P} \Vdash \alpha$ for every statement α in $\mathcal{D}$. $\mathcal{D}$ is said to be *satisfiable* if and only if there is a preferential interpretation that satisfies $\mathcal{D}$.[1]

Definition 7. *A DBox* $\mathcal{D}$ *preferentially entails a defeasible subsumption statement* α, *denoted by* $\mathcal{D} \models_P \alpha$, *if and only if* $\mathcal{P} \Vdash \alpha$ *for every* $\mathcal{P}$ *such that* $\mathcal{P} \Vdash \mathcal{D}$.

One of the ways to evaluate versions of DBox entailment is to consider the defeasible subsumption relation it induces.

[1] It can easily be shown that $\mathcal{D}$ is satisfied in some preferential interpretation if and only if it is satisfied in a ranked interpretation, so this notion of *satisfiability* applies in the context of ranked interpretations as well, a result that we shall use later in this section.

Definition 8. *The set* $\{(C, D) \mid \mathcal{D} \models C \precsim D\}$ *is the subsumption relation induced by a (generic) entailment relation* $\models$ *with respect to a DBox* $\mathcal{D}$*.*

Note that Definition 8 applies to a *generic* entailment relation $\models$ and not just to $\models_P$.

The results below are similar to results obtained in the propositional case [26] and provide strong evidence for the claim that, within the context of preferential interpretations, preferential entailment is *the unique* appropriate version of entailment.

Lemma 3. *If a DBox* $\mathcal{D}$ *is satisfiable, the subsumption relation induced by* $\models_P$ *with respect to* $\mathcal{D}$ *is preferential.*

Further evidence in favor of preferential entailment is obtained by linking it up to the notion of preferential closure. Given a DBox $\mathcal{D}$, we define the *preferential closure* of $\mathcal{D}$ as the intersection of all the preferential subsumption relations containing the set $\{(C, D) \mid C \precsim D \in \mathcal{D}\}$.

Lemma 4. *Let* $\leadsto_{\mathcal{D}}$ *be the (preferential) subsumption relation induced by* $\models_P$ *with respect to a DBox* $\mathcal{D}$*. Then* $\leadsto_{\mathcal{D}}$ *coincides with the preferential closure of* $\mathcal{D}$*.*

Despite these results, it is worth mentioning that $\models_P$ is a *Tarskian consequence relation*, satisfying the following three properties (where $Cn(\mathcal{D}) := \{\alpha \mid \mathcal{D} \models \alpha\}$):

(Inclusion) $\mathcal{D} \subseteq Cn(\mathcal{D})$
(Idempotency) $Cn(\mathcal{D}) = Cn(Cn(\mathcal{D}))$
(Monotonicity) If $\mathcal{D} \subseteq \mathcal{D}'$, then $Cn(\mathcal{D}) \subseteq Cn(\mathcal{D}')$

While Inclusion and Idempotency are desirable properties, the Monotonicity property shows that in spite of the defeasibility features of $\precsim$ we end up with a logic that is *monotonic* at the level of entailment, a somewhat unintuitive result. It can be argued (and has been done in the propositional case [26, 24]) that this is not an indication that preferential entailment is an inappropriate version of entailment *within the context of preferential interpretations*, but rather that the class of preferential interpretations is too weak. It is well-known, for example, that preferential entailment, because of its construction based on preferential interpretations, does not support the inheritance of defeasible properties. This problem carries over to the case for DLs. For example, if we know that both plant cells and mammalian red blood cells are eukaryotic cells ($\mathsf{PlantCell} \sqsubseteq \mathsf{EukCell}$, $\mathsf{MamRBC} \sqsubseteq \mathsf{EukCell}$), that eukaryotic cells usually have a nucleus ($\mathsf{EukCell} \precsim \exists \mathsf{hasNuc}.\top$) and that mammalian red blood cells do not ($\mathsf{MamRBC} \sqsubseteq \neg\exists \mathsf{hasNuc}.\top$), preferential entailment does not allow us to conclude that plant cells usually have a nucleus ($\mathsf{PlantCell} \precsim \exists \mathsf{hasNuc}.\top$).

It is with this criticism in mind that we now shift our attention to DBox entailment based on *ranked* interpretations. The first obvious attempt to do so is to apply Definition 7 to ranked interpretations.

Definition 9. *A defeasible subsumption statement* α *is in the ranked entailment of a DBox* $\mathcal{D}$ *(written as* $\mathcal{D} \models_R \alpha$*) if and only if* $\mathcal{R} \Vdash \alpha$ *for every* $\mathcal{R}$ *such that* $\mathcal{R} \Vdash \mathcal{D}$*.*

It turns out that ranked entailment is problematic for a number of reasons. Firstly, it corresponds *exactly* to preferential entailment, as the following result shows.

Theorem 3. *A subsumption statement α is in the preferential entailment of a DBox $\mathcal{D}$ iff it is in the ranked entailment of $\mathcal{D}$.*[2]

Hence, ranked entailment suffers from exactly the same shortcomings as preferential entailment, including the lack of support for the inheritance of defeasible properties. Related to this is the issue that ranked entailment builds on ranked interpretations, but generates a defeasible subsumption relation that is only preferential. In a sense, a commitment to ranked interpretations implies a commitment to rational subsumption relations (given Theorem 2, which says that the ranked interpretations generate precisely the rational subsumption relations), but ranked entailment violates that commitment.

With the deficiencies of ranked entailment in mind, the goal of the remainder of this section is to define a more appropriate notion of entailment based on ranked interpretations (which we denote by using the $\models$ symbol). We start off with some basic desiderata for this generic $\models$. Our point of departure is the notion of a Tarskian consequence relation (in this case, of course, with $Cn(\mathcal{D}) := \{C \precsim D \mid \mathcal{D} \models C \precsim D\}$.

It seems reasonable to require $\models$ to satisfy Inclusion and Idempotency — two properties associated with Tarskian consequence relations. But as alluded to above, Monotonicity does not seem appropriate. To see why, if we know that mammalian red blood cells are eukaryotic ($\mathsf{MamRBC} \sqsubseteq \mathsf{EukCell}$) and eukaryotic cells usually have a nucleus ($\mathsf{EukCell} \precsim \exists \mathsf{hasNuc}.\top$), then we expect that mammalian red blood cells usually have a nucleus ($\mathsf{MamRBC} \precsim \exists \mathsf{hasNuc}.\top$). But on learning that they do not ($\mathsf{MamRBC} \sqsubseteq \neg\exists \mathsf{hasNuc}.\top$), we would expect the conclusion that they usually have a nucleus to be dropped. We therefore insist on Inclusion and Idempotency, but not on Monotonicity.

The next two requirements relate ranked entailment to $\models$. The first one is based on the idea that, although ranked entailment is too weak, it is a suitable lower bound for $\models$.

(1) If $\mathcal{D} \models_R \alpha$, then $\mathcal{D} \models \alpha$

The second one is based on the idea that ranked entailment deals adequately with *classical* subsumption.

(2) If $\mathcal{D} \models C \sqsubseteq D$, then $\mathcal{D} \models_R C \sqsubseteq D$

For instance, in our running example we expect the classical subsumption statements entailed by the DBox $\{\mathsf{PlantCell} \sqsubseteq \mathsf{EukCell}, \mathsf{MamRBC} \sqsubseteq \mathsf{EukCell}, \mathsf{EukCell} \precsim \exists \mathsf{hasNuc}.\top, \mathsf{MamRBC} \sqsubseteq \neg\exists \mathsf{hasNuc}.\top\}$ to be the classical consequences of $\{\mathsf{PlantCell} \sqsubseteq$

[2] Lehmann and Magidor [26, Section 4.2] proved this for the propositional case, and Britz et al. [10] proved a related result for defeasible modal logics with a semantics different from ours.

$\mathsf{EukCell}$, $\mathsf{MamRBC} \sqsubseteq \mathsf{EukCell}$, $\mathsf{MamRBC} \sqsubseteq \neg\exists \mathsf{hasNuc}.\top\}$, which is what ranked entailment gives us.[3]

The next requirement states the commitment to *rational subsumption relations* discussed earlier.

(3) If $\mathcal{D}$ is satisfiable, then the subsumption relation $\leadsto$ induced by $\models$ w.r.t. $\mathcal{D}$ is a *rational* subsumption relation

Requirement (3) dispenses with a number of problems associated with ranked entailment, including the inheritance of defeasible properties illustrated by our first example about eukaryotic cells.

Given these basic requirements for $\models$, our approach will be to identify a suitable ranked interpretation $\mathcal{R}_{\models}$ as a *canonical* model for defining entailment whenever $\mathcal{D}$ is satisfiable (if $\mathcal{D}$ is unsatisfiable, we have that $\mathcal{D} \models C \sqsubsetapprox D$ for every $C, D \in \mathcal{L}$). That is, we aim to identify an appropriate ranked interpretation $\mathcal{R}_{\models}$ such that we can define $\models$ as follows: $\mathcal{D} \models C \sqsubsetapprox D$ if and only if $\mathcal{R}_{\models} \Vdash C \sqsubsetapprox D$.

Compliance with Requirement (1) is easy to show: If $C \sqsubsetapprox D$ is in the ranked entailment of $\mathcal{D}$, then it is satisfied in every ranked interpretation, and specifically in $\mathcal{R}_{\models}$. It is also easy to see that Requirement (3) is satisfied since we know from Theorem 2 that every $\mathcal{R}$ generates a rational subsumption relation. Given the properties of a rational subsumption relation, it will also satisfy Idempotency. And to ensure that Inclusion is satisfied, we simply require $\mathcal{R}_{\models}$ to satisfy $\mathcal{D}$ (this is possible in all cases except when $\mathcal{D}$ is unsatisfiable).

In order to comply with Requirement (2), we need to define a notion of $\mathcal{D}$-compatibility for classical interpretations. This definition is based on the notion of compatibility defined by Giordano et al. [20] for the propositional case.

Definition 10. *For an interpretation $\mathcal{I}$ and an $x \in \Delta^{\mathcal{I}}$, the tuple $\langle \mathcal{I}, x \rangle$ is $\mathcal{D}$-compatible if and only if $\mathcal{D} \not\models_R C \sqsubsetapprox \bot$ for every $C \in \mathcal{L}$ s.t. $x \in C^{\mathcal{I}}$. $\mathcal{I}$ is said to be $\mathcal{D}$-compatible if and only if for every $x \in \Delta^{\mathcal{I}}$, $\langle \mathcal{I}, x \rangle$ is $\mathcal{D}$-compatible. A ranked interpretation $\mathcal{R}$ is $\mathcal{D}$-compatible if and only if the classical interpretation $\mathcal{I}_{\mathcal{R}}$ associated with it is $\mathcal{D}$-compatible.*

The next result shows that restricting ourselves to $\mathcal{D}$-compatible $\mathcal{R}$s ensures that Requirement (2) is satisfied.

Lemma 5. *For every $\mathcal{D}$-compatible ranked interpretation $\mathcal{R}$, if $\mathcal{R} \Vdash C \sqsubsetapprox \bot$ then $\mathcal{D} \models_R C \sqsubsetapprox \bot$.*

For an interpretation $\mathcal{I}$, let $\mathscr{R}^{\mathcal{I}} = \{\mathcal{R} \mid \Delta^{\mathcal{R}} = \Delta^{\mathcal{I}}$ and $\cdot^{\mathcal{R}} = \cdot^{\mathcal{I}}\}$ be the the set of all ranked interpretations that agree on $\mathcal{I}$. We denote by $\mathscr{R}^{\langle \mathcal{I}, \mathcal{D} \rangle}$ the set of all ranked interpretations in $\mathscr{R}^{\mathcal{I}}$ that satisfy $\mathcal{D}$. If $\mathcal{I}$ is $\mathcal{D}$-compatible, we refer to $\mathscr{R}^{\langle \mathcal{I}, \mathcal{D} \rangle}$ also as $\mathcal{D}$-compatible. The candidates for the canonical model $\mathcal{R}_{\models}$ are precisely the elements of the $\mathcal{D}$-compatible sets $\mathscr{R}^{\langle \mathcal{I}, \mathcal{D} \rangle}$. It is easily shown that every $\mathcal{D}$-compatible $\mathscr{R}^{\langle \mathcal{I}, \mathcal{D} \rangle}$ is non-empty provided that $\mathcal{D}$ is satisfiable. In

[3] This is not to say that defeasible subsumption statements play no part in generating classical subsumption statements. For example $C \sqsubseteq \bot$ is in the ranked entailment of $\mathcal{D} = \{C \sqsubsetapprox D, C \sqsubsetapprox \neg D\}$.

principle all of these ranked interpretations are viable candidates for selection. Indeed, it has been argued in the propositional case [24], as well as in the DL case [15], that there is more than one viable candidate for entailment for defeasible reasoning. Our purpose here is to define the most basic notion of entailment that is appropriate in the context of ranked interpretations.

What follows next is a fundamental result that considerably simplifies the remainder of our task in this section.

Definition 11. *A ranked interpretation $\mathcal{R}$ is finite if and only if there is a mapping r from $\Delta^{\mathcal{R}}$ to a finite set of integers X s.t. for all $x, y \in \Delta^{\mathcal{R}}$, $r(x) < r(y)$ if and only if $x \prec_{\mathcal{R}} y$.*

Lemma 6. *Given a DBox $\mathcal{D}$, for every $\mathcal{D}$-compatible set $\mathscr{R}^{\langle \mathcal{I}, \mathcal{D} \rangle}$ and every $\mathcal{R} \in \mathscr{R}^{\langle \mathcal{I}, \mathcal{D} \rangle}$, there is a* finite *$\mathcal{R}' \in \mathscr{R}^{\langle \mathcal{I}, \mathcal{D} \rangle}$ such that $\mathcal{R} \Vdash \alpha$ if and only if $\mathcal{R}' \Vdash \alpha$ for every defeasible subsumption statement α.*

This ensures that we need only consider the *finite* elements of every $\mathcal{D}$-compatible $\mathscr{R}^{\langle \mathcal{I}, \mathcal{D} \rangle}$, which we denote by $\mathscr{R}_F^{\langle \mathcal{I}, \mathcal{D} \rangle}$. The goal is to place an ordering on the elements of each $\mathcal{D}$-compatible $\mathscr{R}_F^{\langle \mathcal{I}, \mathcal{D} \rangle}$, with ranked interpretations lower down viewed as more basic, and to choose the minimal elements as the candidates for the canonical ranked interpretation.[4]

Definition 12. *For a finite ranked interpretation $\mathcal{R}$ and $x \in \Delta^{\mathcal{R}}$, the height $h_{\mathcal{R}}(x)$ of x is the length of the (finite) chain $x_0 \prec_{\mathcal{R}}, \ldots, \prec_{\mathcal{R}} x$ from x_0 to x, where $x_0 \in \min_{\prec_{\mathcal{R}}}(\Delta^{\mathcal{R}})$.*

For a $\mathcal{D}$-compatible interpretation $\mathcal{I}$, we define the ordering $\leq_{\mathcal{I}}$ on $\mathscr{R}_F^{\langle \mathcal{I}, \mathcal{D} \rangle}$ as follows: $\mathcal{R} \leq_{\mathcal{I}} \mathcal{R}'$ if and only if for every $x \in \Delta^{\mathcal{I}}$, $h_{\mathcal{R}}(x) \leq h_{\mathcal{R}'}(x)$. It is easy to see that $\leq_{\mathcal{I}}$ is a weak partial order and that each $\mathscr{R}_F^{\langle \mathcal{I}, \mathcal{D} \rangle}$ has a (unique) $\leq_{\mathcal{I}}$-minimum element. We are thus left with the choice of one of these $\leq_{\mathcal{I}}$-minimum elements for the canonical model. The next result shows that we can pick any one of them.

Lemma 7. *Let $\mathcal{I}$ and $\mathcal{J}$ be any two $\mathcal{D}$-compatible interpretations. Let $\mathcal{R}^{\mathcal{I}}$ (respectively $\mathcal{R}^{\mathcal{J}}$) be the $\leq_{\mathcal{I}}$-minimum (respectively $\leq_{\mathcal{J}}$-minimum) ranked interpretation in $\mathscr{R}_F^{\langle \mathcal{I}, \mathcal{D} \rangle}$ (respectively $\mathscr{R}_F^{\langle \mathcal{J}, \mathcal{D} \rangle}$). Then $\mathcal{R}^{\mathcal{I}} \Vdash \alpha$ if and only if $\mathcal{R}^{\mathcal{J}} \Vdash \alpha$ for all defeasible subsumption statements α.*

We are now in a position to define an appropriate version of entailment based on ranked interpretations.

Definition 13. *For a satisfiable DBox $\mathcal{D}$, a defeasible subsumption statement α is in the minimal ranked entailment of $\mathcal{D}$ if and only if $\mathcal{R} \Vdash \alpha$, where $\mathcal{R}$ is the $\leq_{\mathcal{I}}$-minimum ranked interpretation of some $\mathcal{D}$-compatible interpretation $\mathcal{I}$. If $\mathcal{D}$ is unsatisfiable then every defeasible subsumption statement α is in the ranked entailment of $\mathcal{D}$.*

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[4] This is based on a similar construction for the propositional case proposed by Giordano et al. [20].

We conclude this section with further evidence that minimal ranked entailment is an appropriate version of entailment by linking it up to the notion of *rational closure*. Initially defined for the propositional case [26], different generalizations of the rational closure were recently provided for DLs [14, 11]. We provided here what we believe to be the most appropriate generalization.

A concept C is said to be *exceptional* for a DBox $\mathcal{D}$ if and only if $\mathcal{D} \models_R \top \mathrel{\sqsubseteq\!\!\!\sim} \neg C$. We use this to build up a sequence of *exceptionality sets* $E_0, E_1, \ldots$, and from this, an *exceptionality ranking* of concepts and defeasible subsumption statements. Let $E(\mathcal{D}) := \{C \mathrel{\sqsubseteq\!\!\!\sim} D \mid \mathcal{D} \models_R \top \mathrel{\sqsubseteq\!\!\!\sim} \neg C\}$. Let $E_0 := \mathcal{D}$, and for $i > 0$, let $E_i := E(E_{i-1})$. Clearly there is a smallest k such that $E_{k+1} = \emptyset$ or $E_k = E_{k+1}$.

Definition 14. *The rank $r_{\mathcal{D}}(C)$ of a concept $C \in \mathcal{L}$ is the smallest number r such that C is not exceptional for E_r. C has rank ∞ if and only if it is exceptional for all E_i (for $i \geq 0$). The rank $r_{\mathcal{D}}(C \mathrel{\sqsubseteq\!\!\!\sim} D)$ of the defeasible subsumption statement $C \mathrel{\sqsubseteq\!\!\!\sim} D$ is the rank $r_{\mathcal{D}}(C)$ of its antecedent C.*

Definition 15. *The rational closure of a DBox $\mathcal{D}$ is the set $\{C \mathrel{\sqsubseteq\!\!\!\sim} D \mid r_{\mathcal{D}}(C) < r_{\mathcal{D}}(C \sqcap \neg D)$ or $r_{\mathcal{D}}(C) = \infty\}$.*

Theorem 4. *For a DBox $\mathcal{D}$, a defeasible subsumption statement α is in the minimal ranked entailment of $\mathcal{D}$ if and only if it is in the rational closure of $\mathcal{D}$.*

5 Computing Minimal Ranked Entailment

Casini and Straccia [14] propose an algorithm for a version of rational closure for $\mathcal{ALC}$ which generalizes the propositional case in a way that is slightly different from Definition 15. The advantage of their approach is that it relies completely on classical $\mathcal{ALC}$-entailment, is easily implementable, and has computational complexity that is no worse than that of classical $\mathcal{ALC}$-entailment. It turns out that a slight modification of their algorithm results in one that corresponds precisely to minimal ranked entailment (and hence to the version of rational closure we defined above). Since in the procedure we are going to present the separation between strict and defeasible knowledge is essential, in the present section we take under consideration knowledge bases $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ that are composed by a classical TBox $\mathcal{T}$ and a DBox $\mathcal{D}$. Since we have shown that every TBox statement can be rewritten as an equivalent DBox statement (Lemma 2), the case in which we start with simply a DBox is just a special case of the following procedure. We shall use the symbol $\models_R^{\leq}$ to indicate the minimal ranked entailment, and the symbol $\vdash_{rat}$ to indicate the inference relation defined by the procedure we are going to present. The aims of this section are i) to present the decision procedure defining $\vdash_{rat}$, and ii) to prove the correspondence between $\models_R^{\leq}$ and $\vdash_{rat}$.

5.1 The inference relation $\vdash_{rat}$.

Consider a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$, with a finite TBox $\mathcal{T} = \{E_1 \sqsubseteq F_1, \ldots, E_m \sqsubseteq F_m\}$ and a finite DBox $\mathcal{D} = \{C_1 \mathrel{\sqsubseteq\!\!\!\!\sim} D_1, \ldots, C_n \mathrel{\sqsubseteq\!\!\!\!\sim} D_n\}$. The procedure to define the inference relation $\vdash_{rat}$ is the following:

Step 1. Set $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{D}^0 = \mathcal{D}$ and $i = 0$. Repeat **Steps 1.1** and **1.2** until $\mathcal{D}^i_\infty = \emptyset$.

Step 1.1. Let $\overline{\mathcal{D}^i}$ be the set containing the *materializations* of the axioms in $\mathcal{D}^i$, i.e. $\overline{\mathcal{D}^i} = \{\neg C \sqcup D \mid C \mathrel{\sqsubseteq\!\!\!\!\sim} D \in \mathcal{D}\}$; by *materialization* of an inclusion axiom $C \mathrel{\sqsubseteq\!\!\!\!\sim} D$ we indicate the concept that expresses in the language the same inclusion relation as the one expressed by the axiom (if an object falls under C, then it falls also under D). Moreover, let $\mathfrak{A}_{\mathcal{D}^i}$ be the set of the antecedents of the conditionals in $\mathcal{D}^i$ ($\mathfrak{A}_{\mathcal{D}^i} = \{C \mid C \mathrel{\sqsubseteq\!\!\!\!\sim} D \in \mathcal{D}^i\}$).
We determine an *exceptionality ranking* of the axioms in $\mathcal{D}^i$ using $\mathfrak{A}_{\mathcal{D}^i}$, $\mathcal{T}^i$ and $\overline{\mathcal{D}^i}$.
A concept is considered *exceptional* in a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$ if

$$\mathcal{T} \models \bigsqcap \overline{\mathcal{D}} \sqsubseteq \neg C.$$

If a concept C is exceptional in $\langle \mathcal{T}, \mathcal{D} \rangle$, also all the defeasible inclusion axioms having C as antecedent are considered exceptional. Given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$, we can define a function $E_{\mathcal{T}}$ that gives back exceptional axioms in $\mathcal{D}$ ($E_{\mathcal{T}}(\mathcal{D}) = \{C \mathrel{\sqsubseteq\!\!\!\!\sim} D \in \mathcal{D} \mid \ \mathcal{T}^i \models \bigsqcap \overline{\mathcal{D}} \sqsubseteq \neg C\}$).
Given $\langle \mathcal{T}^i, \mathcal{D}^i \rangle$ we can construct a sequence $\mathcal{E}_0, \mathcal{E}_1, \ldots$ in the following way:

- $\mathcal{E}_0 = \mathcal{D}^i$
- $\mathcal{E}_{j+1} = E_{\mathcal{T}}(\mathcal{E}_j)$

Since $\mathcal{D}^i$ is a finite set, the construction will terminate with a (empty or not-empty) fixed point of $E_{\mathcal{T}}$.

Step 1.2. We define a ranking function r^i that associates to every axiom in $\mathcal{D}^i$ a number, representing its level of exceptionality:

$$r^i(C \mathrel{\sqsubseteq\!\!\!\!\sim} D) = \begin{cases} j & \text{if } C \mathrel{\sqsubseteq\!\!\!\!\sim} D \in \mathcal{E}_j \text{ and } C \mathrel{\sqsubseteq\!\!\!\!\sim} D \notin \mathcal{E}_{j+1} \\ \infty & \text{if } C \mathrel{\sqsubseteq\!\!\!\!\sim} D \in \mathcal{E}_j \text{ for every } j\ . \end{cases}$$

And the same applies to the concepts appearing as antecedents in the axioms in $\mathcal{D}$:

$$r^i(C) = r^i(C \mathrel{\sqsubseteq\!\!\!\!\sim} D) \text{ for every } C \mathrel{\sqsubseteq\!\!\!\!\sim} D \in \mathcal{D}^i$$

We indicate with $\mathcal{D}^i_j$ the set of the defeasible axioms in $\langle \mathcal{T}^i, \mathcal{D}^i \rangle$ having j as ranking value. Hence the set $\mathcal{D}^i$ is partitioned into the sets $\mathcal{D}^i_0, \ldots, \mathcal{D}^i_n, \mathcal{D}^i_\infty$, for some n, and with $\mathcal{D}^i_\infty$ possibly empty. Let us call $\mathcal{D}^i_{\sqsubseteq}$ the set of strict axioms corresponding to $\mathcal{D}^i_\infty$, that is, $\mathcal{D}^i_{\sqsubseteq} = \{C \sqsubseteq \bot \mid C \mathrel{\sqsubseteq\!\!\!\!\sim} D \in \mathcal{D}^i_\infty\}$. We define a new knowledge base $\langle \mathcal{T}^{i+1}, \mathcal{D}^{i+1} \rangle$, with

- $\mathcal{T}^{i+1} = \mathcal{T}^i \cup \mathcal{D}^i_{\sqsubseteq}$, and
- $\mathcal{D}^{i+1} = \mathcal{D}^i / \mathcal{D}^i_\infty$.

The only difference between $\langle \mathcal{T}^i, \mathcal{D}^i \rangle$ and $\langle \mathcal{T}^{i+1}, \mathcal{D}^{i+1} \rangle$ is that some classical knowledge that was 'implicitly contained' in $\mathcal{D}^i$ is now moved into $\mathcal{T}^{i+1}$, and the set $\mathcal{D}^{i+1}$ is partitioned by the ranking function r^i into $\mathcal{D}_0^i, \ldots, \mathcal{D}_n^i$.
Set $i = i + 1$.

Step 2. Since $\mathcal{D}$ is a finite set and $\mathcal{D}^{i+1} \subseteq \mathcal{D}$, after some iterations of Steps 1.1 and 1.2 we necessarily obtain $\mathcal{D}_\infty^n = \emptyset$ for some $n \leq |\mathcal{D}|$, and the ranking procedure (**Steps 1.1** and **1.2**) stops, since all the strict information contained in $\mathcal{D}$ has been moved into the Tbox. Set $\mathcal{T}' = \mathcal{T}^n$, $\mathcal{D}' = \mathcal{D}^n$, and $r' = r^n$.
Once we have defined $\langle \mathcal{T}', \mathcal{D}' \rangle$, let Δ be a sequence of concepts $\langle \delta_0, \ldots, \delta_n \rangle$ s.t. $\delta_i = \bigsqcap \{\neg C \sqcup D \mid C \precsim D \in \mathcal{D}' \text{ and } i \leq r'(C \precsim D)\} \leq n\}$; note that if $j < i$, then $\models \delta_j \sqsubseteq \delta_i$.
We can extend the definition of r' to all the concepts and all the defeasible axioms in our language in the following way. For every concepts C and D:

- $r'(C) = i$, $0 \leq i \leq n$, if δ_i is the first element in $\langle \delta_0, \ldots, \delta_n \rangle$ s.t. $\mathcal{T}' \not\models \delta_i \sqcap C \sqsubseteq \bot$.
- $r'(C) = \infty$ if there is no such δ_i.
- $r'(C \precsim D) = r'(C)$

It is immediate to see that for every C s.t. $C \precsim D \in \mathcal{D}'$, this definition of r' gives exactly the same result as the r' defined in the above procedure.
We shall see in Corollary 1 that the two knowledge bases $\langle \mathcal{T}, \mathcal{D} \rangle$ and $\langle \mathcal{T}', \mathcal{D}' \rangle$ are preferentially equivalent.

Step 3. Now, given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$ and obtained the equivalent knowledge base $\langle \mathcal{T}', \mathcal{D}' \rangle$,we can define the inference relation $\vdash_{rat}$.

Definition 16 (Inference relation $\vdash_{rat}$). *$\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \precsim D$ iff $\mathcal{T}' \models \delta_i \sqcap C \sqsubseteq D$, where δ_i is the first element of the sequence $\langle \delta_0, \ldots, \delta_n \rangle$ s.t. $\mathcal{T}' \not\models \delta_i \sqsubseteq \neg C$; if there is no such element, $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \precsim D$ iff $\mathcal{T}' \models C \sqsubseteq D$.*
$\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \sqsubseteq D$ iff $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \sqcap \neg D \precsim \bot$, i.e., iff $\mathcal{T}' \models C \sqcap D \sqsubseteq \bot$ (that is to say $\mathcal{T}' \models C \sqsubseteq D$).

Let us call C_{rat} the closure operation corresponding to $\vdash_{rat}$, i.e., $C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle) = \{C \precsim D \mid \langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \precsim D\} \cup \{C \sqsubseteq D \mid \langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \sqcap \neg D \precsim \bot\}$. We can prove that $\mathcal{T}$ and $\mathcal{D}$ are in $C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ and that $C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ defines a rational subsumption relation.

Proposition 1. *Given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$, $\mathcal{T} \cup \mathcal{D} \in C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$. $C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ defines a defeasible subsumption relation $\precsim$ that is rational.*

Proof. Assume that $C \sqsubseteq D \in \mathcal{T}$. $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \sqsubseteq D$ iff $\mathcal{T}' \models C \sqsubseteq D$; since $\mathcal{T} \subseteq \mathcal{T}'$, $\mathcal{T} \subseteq C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$.

Assume that $C \precsim D \in \mathcal{D}$. Either $C \precsim D \in \mathcal{D}_\infty^j$ for some j, or there will be an i ($0 \leq i \leq n$) s.t. $r'(C) = r'(C \precsim D) = i$. In the former case, $C \sqsubseteq \bot$ is in $\mathcal{T}'$, and so $\mathcal{T}' \models C \sqsubseteq D$, i.e., $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \precsim D$.

In the latter case, $\models \delta_i \sqsubseteq \neg C \sqcup D$, and so $\mathcal{T}' \models \delta_i \sqcap C \sqsubseteq D$, i.e., $C \precsim D \in C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$. Hence $\mathcal{T} \cup \mathcal{D} \subseteq C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$.

Given Definition 16, to prove that $C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ satisfies the rational properties w.r.t. $\precsim$ is quite straightforward.

- (Ref). Since $\models C \sqsubseteq C$ is valid in $\mathcal{ALC}$ for any C, we have that $\mathcal{T}' \models \delta_i \sqcap C \sqsubseteq C$ for any $\mathcal{T}'$ and δ_i.
- (LLE). $C \precsim E \in C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ implies that $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq E$ for some δ_i (or $\mathcal{T}' \models C \sqsubseteq E$, if $r'(C) = \infty$). Since $\models C \equiv D$, $\mathcal{T}' \models \delta i \sqcap D \sqsubseteq E$ too.
- (And). $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq D$ and $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq E$ (possibly without the δ_i, if C has infinite ranking), hence $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq D \sqcap E$, that is, $C \precsim D \sqcap E \in C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$.
- (Or). $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq E$ for some δ_i and $\mathcal{T}' \models \delta j \sqcap D \sqsubseteq E$ for some δ_j. Let's assume that $i < j$, that is, $\models \delta_i \sqsubseteq \delta_j$. Then, since $\mathcal{T}' \not\models \delta_i \sqsubseteq \neg C$, we have that $\mathcal{T}' \not\models \delta_i \sqsubseteq \neg(C \sqcup D)$. Moreover $\mathcal{T}' \models \delta j \sqcap D \sqsubseteq E$ and $\models \delta_i \sqsubseteq \delta_j$ imply that $\mathcal{T}' \models \delta i \sqcap D \sqsubseteq E$. So, $\mathcal{T}' \models \delta i \sqcap (C \sqcup D) \sqsubseteq E$. The proof is analogous for $j < i$, or if i or j corresponds to ∞.
- (RW). $C \precsim D \in C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ if $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq D$ for some δ_i (or $\mathcal{T}' \models C \sqsubseteq D$, if $r'(C) = \infty$). Since $\models D \sqsubseteq E$, $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq E$.
- (CM). If $r'(C) = i < \infty$, $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq D$ and $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq E$ for some δ_i. Since $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq D$ and $\mathcal{T}' \not\models \delta i \sqsubseteq \neg C$, $\mathcal{T}' \not\models \delta i \sqsubseteq \neg(C \sqcap D)$, otherwise we would have $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq D \sqcap \neg D$, i.e., $\mathcal{T}' \models \delta i \sqsubseteq \neg C$. Hence we have $C \sqcap D \precsim E \in C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ since $\mathcal{T}' \models \delta i \sqcap C \sqcap D \sqsubseteq E$. If $r'(C) = \infty$, we have $\mathcal{T}' \models C \sqsubseteq \bot$, and the proof is trivial.
- (RM). If $r'(C) = i < \infty$, $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq E$ and $\mathcal{T}' \not\models \delta i \sqcap C \sqsubseteq \neg D$ for some δ_i. Since $\mathcal{T}' \not\models \delta i \sqcap C \sqsubseteq \neg D$ and $\mathcal{T}' \not\models \delta i \sqsubseteq \neg C$, $\mathcal{T}' \not\models \delta i \sqsubseteq \neg(C \sqcap D)$, otherwise we would have $\mathcal{T}' \models \delta i \sqcap C \sqsubseteq \neg D$. Hence we have $C \sqcap D \precsim E \in C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ since $\mathcal{T}' \models \delta i \sqcap C \sqcap D \sqsubseteq E$. If $r'(C) = \infty$, we have $\mathcal{T}' \models C \sqsubseteq \bot$, and the proof is trivial.

The preferential entailment satisfies also two stronger form of (LLE) and (RW), i.e., (LLE') and (RW'), that have the following form:

$$\text{(RW')}\ \frac{\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} C \precsim D,\ \mathcal{T} \models D \sqsubseteq E}{\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} C \precsim E} \qquad \text{(LLE')}\ \frac{\mathcal{T} \models C \equiv D,\ \langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} C \precsim E}{\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} D \precsim E}$$

The proofs are quite straightforward, since all the preferential models satisfying $\langle \mathcal{T}, \mathcal{D} \rangle$ must satisfy $\mathcal{T}$. The validity of the analogous properties w.r.t. $\vdash_{rat}$ is provable using proofs that are analogous to the ones for (RW) and (LLE).

5.2 Correspondence between $\models_R^{\leq}$ and $\vdash_{rat}$.

We want to prove that the procedure above defining $\vdash_{rat}$ corresponds to the notion of minimal ranked entailment $\models_R^{\leq}$. First of all, we need to prove that $\langle \mathcal{T}, \mathcal{D} \rangle$ and $\langle \mathcal{T}', \mathcal{D}' \rangle$ are preferentially equivalent.

The following lemma proves that our procedure manages correctly the strict information.

Lemma 8. *Assume that $C \precsim D \in \mathcal{D}$. $\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} C \precsim \bot$ if and only if $r'(C) = \infty$, i.e., if and only if $\mathcal{T}' \models C \sqsubseteq \bot$.*

Proof. $\Rightarrow$: $\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} C \precsim \bot$ implies that every preferential consequence relation satisfying $\langle \mathcal{T}, \mathcal{D} \rangle$ satisfies also $C \precsim \bot$, hence we have that $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \precsim \bot$, since $C_{rat}(\langle \mathcal{T}, \mathcal{D} \rangle)$ defines a preferential consequence relation satisfying $\langle \mathcal{T}, \mathcal{D} \rangle$ (Proposition 1). From Definition 16 we know that $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \precsim \bot$ is possible only if C is always negated in the ranking procedure, i.e., $\mathcal{T}' \models C \sqsubseteq \bot$.

$\Leftarrow$: If $\mathcal{D}_\infty = \emptyset$ the result is immediate, since $\mathcal{T} \models C \sqsubseteq \bot$ must be the case, hence consider the case that $\mathcal{D}_\infty \neq \emptyset$ and we define a new knowledge base $\langle \mathcal{T}', \mathcal{D}' \rangle$, with $\mathcal{T}' = \mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^n_\sqsubseteq$. Assume that $\mathcal{T}' \models C \sqsubseteq \bot$, but $\langle \mathcal{T}, \mathcal{D} \rangle \not\models_{\mathcal{P}} C \precsim \bot$, i.e., there is a preferential model of $\langle \mathcal{T}, \mathcal{D} \rangle$ s.t. C is non-empty. Consider such a model $\mathcal{M}$, with an object a falling under $C^{\mathcal{I}}$. Since $\mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^n_\sqsubseteq \models C \sqsubseteq \bot$, there must be a subsumption axiom $E \sqsubseteq \bot$ in some $\mathcal{D}^i_\sqsubseteq$ that is not satisfied, hence there must be an individual b falling under E in $\mathcal{M}$. Hence, since $\mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^{i-1}_\sqsubseteq \models \overline{\mathcal{D}^i_\infty} \sqsubseteq \neg E$, either $\mathcal{M} \Vdash \mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^{i-1}_\sqsubseteq$ and $\mathcal{M} \Vdash F \sqcap \neg G(b)$ for some $F \precsim G \in \mathcal{D}^i_\infty$ (Case 1), or $\mathcal{M} \not\Vdash \mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^{i-1}_\sqsubseteq$ (Case 2).

Case 1. Since $\mathcal{M}$ is a model of $\langle \mathcal{T}, \mathcal{D} \rangle$, hence it is a model also of $F \precsim G$, that is in $\mathcal{D}$. Hence there must be an individual c s.t. $c \prec b$ and $F \sqcap G(c)$. Again, since $F \precsim G \in \mathcal{D}^i_\infty$ (i.e., $\mathcal{T} \cup \mathcal{D}^0_\sqsubseteq \cup \ldots \cup \mathcal{D}^{i-1}_\sqsubseteq \models \overline{\mathcal{D}^i_\infty} \sqsubseteq \neg F$) and $\mathcal{M} \Vdash \mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^{i-1}_\sqsubseteq$, there must be an axiom $H \precsim I \in \mathcal{D}^i_\infty$ s.t. $\mathcal{M} \vdash H \sqcap \neg I(c)$, and we need an individual d s.t. $d \prec c$ and $\mathcal{M} \vdash H \sqcap I(d)\ldots$
This procedure creates an infinite descending chain of individuals, and, since the number of the antecedents of the axioms in $\mathcal{D}^i_\infty$ is finite, it cannot be the case since the model would not satisfy the smoothness condition for the concept $\bigsqcup\{C | C \precsim D \in \mathcal{D}^i_\infty\}$.

Case 2. If $\mathcal{M} \not\Vdash \mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^{i-1}_\sqsubseteq$, then $\mathcal{M}$ does not satisfy some axiom $E \sqsubseteq \bot \in \mathcal{D}^j_\sqsubseteq$ for some $j < i$, and there must be an oboect falling under E in $\mathcal{M}$. Again, it is Case 1 or Case 2. However, since at every reiteration of Case 2 we pick a lower value j for $\mathcal{D}^j_\sqsubseteq$ and we have a finite sequence of $\mathcal{D}^j_\sqsubseteq$, we know that after some steps (in the worst case when we reach $\mathcal{D}^0_\sqsubseteq$) we necessarily fall into Case 1, that cannot be the case.

An immediate result of Lemma 8 binds preferential consistency to classical entailment.

Lemma 9. $\mathcal{T}' \models \top \sqsubseteq \bot$ *if and only if* $\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} \top \precsim \bot$.

We can now prove that the knowledge bases $\langle \mathcal{T}, \mathcal{D} \rangle$ and $\langle \mathcal{T}', \mathcal{D}' \rangle$ are preferentially equivalent.

Corollary 1. *Consider a KB* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ *and another KB* $\mathcal{K}' = \langle \mathcal{T}', \mathcal{D}' \rangle$ *obtained from* $\mathcal{K}$ *using the procedure explained in the Steps 1.1-1.2 in the above definition of* $\vdash_{rat}$. $\mathcal{K}$ *and* $\mathcal{K}'$ *are preferentially equivalent.*

Proof. We have $\mathcal{T}' = \mathcal{T} \cup \mathcal{D}^1_\sqsubseteq \cup \ldots \cup \mathcal{D}^n_\sqsubseteq$ and $\mathcal{D}' = \mathcal{D} / (\mathcal{D}^1_\infty \cup \ldots \cup \mathcal{D}^n_\infty)$. It is sufficient to prove that $\mathcal{K} \models_{\mathcal{P}} C \sqsubseteq \bot$ for every $C \precsim D \in \mathcal{D}^i_\infty$, for $1 \leq i \leq n$, and $\mathcal{K}' \models_{\mathcal{P}} C \precsim D$ for every $C \precsim D \in \mathcal{D}^i_\infty$, for $1 \leq i \leq n$.

If $C \precsim D \in \mathcal{D}_\infty^i$, for $1 \leq i \leq n$, then $C \sqsubseteq \bot \in \mathcal{T}'$, and $\mathcal{T}' \models C \sqsubseteq \bot$ obviously; by Lemma 8 we have that $\mathcal{K} \models_{\mathcal{P}} C \precsim \bot$, i.e., $\mathcal{K} \models_{\mathcal{P}} C \sqsubseteq \bot$.

On the other hand, if $C \precsim D \in \mathcal{D}_\infty^i$, for $1 \leq i \leq n$, $C \sqsubseteq \bot \in \mathcal{T}'$, and hence $\mathcal{K}' \models_{\mathcal{P}} C \precsim D$ by supraclassicality (see Lemma 10 below) and RW.

Now we are justified in using the knowledge base $\langle \mathcal{T}', \mathcal{D}' \rangle$ in order to analyse the rational closure of the knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$. In particular, now we can check that the inference relation $\vdash_{rat}$ respects the preferential conclusions of $\langle \mathcal{T}, \mathcal{D} \rangle$ w.r.t. the assertions of form $\top \precsim C$.

Lemma 10. *$\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} \top \precsim C$ if and only if $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} \top \precsim C$.*

Proof. Remember that $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} \top \precsim C$ iff $\mathcal{T}' \models \bigsqcap \overline{\mathcal{D}'} \sqsubseteq C$.

$\Leftarrow$: First, we need to prove two properties of $\models_{\mathcal{P}}$: supraclassicality (Sup) and one half of the deduction theorem (S):

(Sup):

$$\frac{C \sqsubseteq D}{C \precsim D}$$

Assume $C \precsim C$ and $C \sqsubseteq D$ and apply (RW).

(S):

$$\frac{C \precsim D}{\top \precsim \neg C \sqcup D}$$

Assume $C \precsim D$ and $\models D \sqsubseteq \neg C \sqcup D$ (that is classically valid); we derive by (RW) $C \precsim \neg C \sqcup D$. Assume $\models \neg C \sqsubseteq \neg C \sqcup D$ (classically valid); we obtain $\neg C \precsim \neg C \sqcup D$ by (Sup). We apply (Or) to $C \precsim \neg C \sqcup D$ and $\neg C \precsim \neg C \sqcup D$, obtaining $\top \precsim \neg C \sqcup D$.

Now we have to prove that if $\mathcal{T}' \models \bigsqcap \overline{\mathcal{D}'} \sqsubseteq C$, then $\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} \top \precsim C$.

From corollary 1 we know that $T' \cup \mathcal{D}'$ is in the preferential consequences of $\langle \mathcal{T}, \mathcal{D} \rangle$. Applying (S) to all the axioms $C \precsim D$ in $\mathcal{D}'$, and we have $\langle \mathcal{T}, \mathcal{D} \rangle \models_{\mathcal{P}} \top \precsim \neg C \sqcup D$ from each of them. Applying (And) to all these defeasible inclusions, we have $\top \precsim \bigsqcap \overline{\mathcal{D}'}$ and, by (RW'), we obtain $\top \precsim C$.

$\Rightarrow$: Immediate from Proposition 1.

Now, we want to prove that, given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$, $\langle \mathcal{T}, \mathcal{D} \rangle \models_R^{\leq} C \precsim D$ if and only if $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \precsim D$.

In what follows (specifically, in the proof of Lemma 10) we need to assume that the knowledge bases $\langle \mathcal{T}, \mathcal{D} \rangle$ we are working with are such that they do not have an infinite ranking, that is $\mathcal{D}_\infty = \emptyset$; Corollary 1 allows us to do such an assumption without any harm, since we can assume that our knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$ has already been transformed into the preferentially equivalent knowledge base $\langle \mathcal{T}', \mathcal{D}' \rangle$.

Now we begin to check the correspondence between the ranking function presented in the paper ($r_{\mathcal{D}}$, or, in the present more general form, $r_{\langle \mathcal{T}, \mathcal{D} \rangle}$) and the ranking function defined in the above procedure (r').

Corollary 2. *For every $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, $r_{\langle \mathcal{T}, \mathcal{D} \rangle}(C) = \infty$ iff $r'(C) = \infty$.*

Proof. Consider a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, that, as explained in the paper (Lemma 2), is transformed in a preferentially equivalent knowledge base $\mathcal{D}'$, composed of only defeasible subsumption statements. Since the minimal ranked model of $\mathcal{K}$ is a $\mathcal{D}'$-compatible ranked interpretation, we can easily derive from Lemma 5 and Theorem 3 in the paper that $\mathcal{K} \models_R^{\leq} C \sqsubsetsim \bot$ iff $\mathcal{K} \models_{\mathcal{P}} C \sqsubsetsim \bot$. Moreover, from Definition 15 in the paper we can see that $\mathcal{K} \models_R^{\leq} C \sqsubsetsim \bot$ iff $r_{\mathcal{D}}(C) = \infty$, and from Lemma 8 we have that $\mathcal{K} \models_{\mathcal{P}} C \sqsubsetsim \bot$ iff $r'(C) = \infty$, hence the result.

Now, the main result. First, we need to prove that the two ranking functions correspond to each other.

Proposition 2. *Given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$,the function $r_{\langle \mathcal{T}, \mathcal{D} \rangle}$ and the function r' define the same ranking of the axioms in $\mathcal{D}$.*

Proof. From Lemma 8, Corollary 2 and Corollary 1 we can see that, given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$ (possibly, with an empty $\mathcal{T}$), we can define a preferentially equivalent knowledge base $\langle \mathcal{T}', \mathcal{D}' \rangle$ s.t. all the classical information implicit in $\mathcal{D}$ is made explicit into $\mathcal{T}'$. $\langle \mathcal{T}', \mathcal{D}' \rangle$ can be defined identifying the elements of $\mathcal{D}$ that have ∞ as ranking value, and Corollary 2 shows that w.r.t. the value ∞, $r_{\langle \mathcal{T}, \mathcal{D} \rangle}$ and r' are equivalent, while Corollary 1 tells us that $\langle \mathcal{T}, \mathcal{D} \rangle$ and $\langle \mathcal{T}', \mathcal{D}' \rangle$ are preferentially equivalent. Once we have defined $\langle \mathcal{T}', \mathcal{D}' \rangle$, Lemma 10 implies that a concept C is exceptional w.r.t. $\models_R^{\leq}$ iff it is exceptional w.r.t. $\vdash_{rat}$. Hence the two ranking functions $r_{\langle \mathcal{T}, \mathcal{D} \rangle}$ and r' give back exactly the same results.

Now we can prove the main theorem. The following theorem is clearly a generalization of Theorem 5 in the paper.

Theorem 5. *Given a knowledge base $\langle \mathcal{T}, \mathcal{D} \rangle$, $\langle \mathcal{T}, \mathcal{D} \rangle \models_R^{\leq} C \sqsubsetsim D$ iff $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \sqsubsetsim D$.*

Proof. Since we have proved Proposition 2, in this proof we are generically going to use r to indicate indifferently the equivalent ranking functions $r_{\langle \mathcal{T}, \mathcal{D} \rangle}$ and r'.

$\Rightarrow$: Assume $\langle \mathcal{T}, \mathcal{D} \rangle \models_R^{\leq} C \sqsubsetsim D$. That means that either $r(C \sqcap \neg D) > r(C)$ or $r(C) = \infty$. In the first case, by the definition of $\vdash_{rat}$ we know that there is an i, $0 \leq i \leq n$, s.t. $\mathcal{T}' \not\models \delta_i \sqsubseteq \neg C$ and $\mathcal{T}' \models \delta_i \sqsubseteq \neg(C \sqcap \neg D)$, hence $\mathcal{T}' \models \delta_i \sqcap C \sqsubseteq D$, i.e., $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \sqsubsetsim D$. In the second case, we have $\mathcal{T}' \models C \sqsubseteq \bot$, that implies $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \sqsubsetsim D$.

$\Leftarrow$: Assume $\langle \mathcal{T}, \mathcal{D} \rangle \vdash_{rat} C \sqsubsetsim D$. Then either there is an i which is the lowest number s.t. $\mathcal{T}' \not\models \delta_i \sqsubseteq \neg C$ (hence $r(C) = i$), or $\mathcal{T}' \models C \sqsubseteq \bot$. In the first case we have also that $\mathcal{T}' \models \delta_i \sqcap C \sqsubseteq D$, that implies that $\mathcal{T}' \models \delta_i \sqsubseteq \neg(C \sqcap \neg D)$, i.e. $r(C \sqcap \neg D) > i$. In the second case, $r(C) = \infty$, i.e., $\langle \mathcal{T}, \mathcal{D} \rangle \models_R^{\leq} C \sqsubsetsim D$.

6 Discussion and Related Work

Closely related to our work is that of Giordano et al. [19, 20] which uses preferential orderings on $\Delta^{\mathcal{I}}$ to define a typicality operator T s.t. the expression

$\mathsf{T(C)} \sqsubseteq \mathsf{D}$ corresponds to our $C \precsim D$. They provide a tableaux calculus for their system that relies on the KLM-style rules. However, they have not developed the main representation results, and in order to augment the inferential power of their system they have used circumscription techniques, obtaining systems that share properties of both the KLM and the circumscription approaches. Casini and Straccia [14, 29] present KLM-based decision procedures for $\mathcal{ALC}$. Their proposal has a syntactic characterization, but lacks an appropriate semantics, a deficiency that this paper remedies.

Outside the family of preferential systems there are mature proposals based on circumscription for DLs [6, 5, 4, 28]. The main drawback of these approaches is the burden on the ontology engineer to make appropriate decisions related to the (circumscriptive) fixing and varying of concepts and the priority of defeasible subsumption statements. Such choices can have a major effect on the conclusions drawn from the system, and can easily lead to counter-intuitive conclusions. Moreover, the use of circumscription usually implies a considerable increase in computational complexity w.r.t. the underlying monotonic entailment relation. The comparison between the present work and proposals outside the preferential family is more an issue about the pros and cons of the different kinds of non-monotonic reasoning, rather than about their DL re-formulation. As stated in the introduction, the preferential approach has a series of desirable qualities that, to our knowledge, no other approach to non-monotonic reasoning shares.

Britz and Varzinczak [12, 13] explore the notion of *defeasible modalities*, with which defeasible effects of actions, defeasible knowledge, obligations and others can be formalized. Their approach differs from ours in that it is only preferential, but the semantic constructions are. Inspired by their definitions, we can explore a notion of *defeasible role restrictions* in a DL setting. The idea comprises extending the language of $\mathcal{ALC}$ with an additional construct $\forall\!\!\!\sim$. The semantics of a concept $\forall\!\!\!\sim r.C := \{x \in \Delta^{\mathcal{P}} \mid \min_{\prec_{\mathcal{P}}}(r^{\mathcal{P}}(x)) \subseteq C^{\mathcal{P}}\}$ is then given by all objects of $\Delta^{\mathcal{P}}$ such that all of their *minimal* r-related objects are C-instances. This becomes particularly useful in modeling concepts in domains such as chemical process engineering, where we want to refer to the normality of a process modeled as a role. For example, the subsumption statement $\mathsf{Feed} \sqsubseteq \forall\!\!\!\sim\mathsf{filtration.Fluid}$ models the fact that a feed (always) delivers (only) a fluid as normal outcome of filtration. It is not the normality of the feed or the fluid which is at stake here, but rather the normality of the filtration process.

7 Concluding Remarks

The contributions of this paper are as follows: (*i*) The provision of a simple and intuitive semantics for defeasible subsumption in description logics that is general enough to constitute the core framework within which to investigate non-monotonic extensions of DLs; (*ii*) A characterization of preferential and rational subsumption relations, with the respective representation results; (*iii*) An analysis of what an appropriate notion of entailment in a defeasible DL context mean and the identification of a suitable candidate, namely minimal ranked entailment, and (*iv*) The formal connection between our entailment, the notion of rational closure and a syntactic method for its computation.

A topic for further research is the integration of notions such as *typicality* [19, 7] and the aforementioned defeasible role restrictions into the framework here presented. Another avenue for future exploration is the study of belief revision for DLs via our results for rationality, mimicking the well-known link between belief revision and rational consequence in the propositional case [18], thereby pushing the frontiers of defeasible reasoning in logics that are more expressive than the propositional one.

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